of the circles M and N, go into A' and B', intersections of M' a diameter of K', and N' a circle diametric to K'. Hence A' and B' are inverse with respect to K'.

This answers question (2). And question (1) may then be answered in the affirmative in view of the two following facts:

First, any arbitrary finite group of fractional linear substitutions may be obtained by transforming one of Klein's canonical groups by a fractional linear substitution; and

Secondly, if the poles of a fractional linear substitution are skew-inverse with respect to some circle C, then this substitution corresponds, by stereographic projection, to a rotation of the sphere which has C as a great circle.

HISTORICAL NOTE ON THE NEWTON-RAPHSON METHOD OF APPROXIMATION.

By FLORIAN CAJORI, Colorado College.

Newton explained his method of approximation to the real roots of numerical equations in a tract, *De analysi per aequationes numero terminoraum infinitas*,* which is celebrated chiefly as containing the first announcement of the principle of fluxions and the binomial theorem. Newton placed it in the hands of his teacher, Isaac Barrow, in 1669. Barrow sent it to John Collins, a member of the Royal Society, who, for his zeal for collecting and diffusing scientific information, received the sobriquet of "English Mersenne." The tract became known to some correspondents of Collins and to friends of Newton, but it was not printed until 1704 and 1711. Substantially the same explanation of his method of approximation to the roots of numerical equations was given by Newton in a second tract, the *Methodus fluxionum et serierum infinitarum*, which was planned for publication in 1671, but was not printed until 1736.

The earliest printed account of Newton's method of approximation appeared in Wallis' *Algebra*, London, 1685, chapter 94. Wallis explains Newton's process of solving $y^3-2y-5=0$, and derives, in the equation $y^3+axy+aay-x^3-2a^3=0$, Newton's value of y expressed in terms of a and x by a rapidly converging series. The two equations are treated by Newton in much the same manner. The solution of $y^3-2y-5=0$ is exhibited in Wallis' *Algebra*, as also in the two tracts of Newton referred to above, in paradigms which are identical except in some of the last digits in the decimal fractions. The following is copied from page 268 of Vol. I of Newton's *Opera* (Ed. Horsley):

* Isaac Newton's *Opera* (Edition by Horsley), 1779-1785, Vol. I, pp. 268-269.

$y^3-2y-5=0$		$+2.10000000$ -0.00544853 $+2.09455147=y$
$2+p=y$	$+y^3$ $-2y$ -5	$+8+12p+6p^2+p^3$ $-4-2p$ -5
	Summa	$-1+10p+6p^2+p^3$
$0,1+q=p$	$+p^3$ $+6p^2$ $+10p$ -1	$+0,001+0,03q+0,3q^2+q^3$ $+0,06\ +1,2\ +6,0$ $+1\ +10,$ $-1,$
	Summa	$+0,061+11,23q+6,3q^2+q^3$
$-0,0054+r=q$	$+6,3q^2$ $+11,23q$ $+0,061$	$+0,000183708-0,06804r+6,3r^2$ $-0,060642\ +11,23$ $+0,061$
	Summa	$+0,000541708+11,16196r+6,3r^2$
$-0,00004854+s=r$		

We see that Newton takes as the initial approximation to the real root $y=2$. He expresses himself thus: "Sit 2 numerus qui minùs quàm decimâ sui parte differt à radice quaesitâ. Tum pono $2+p=y$..." The equation in p becomes $p^3+6p^2+10p-1=0$. Neglecting the higher powers of p, he gets $10p-1=0$; taking $p=.1+q$, he gets $q^3+6.3q^2+11.23q+.061=0$. From $11.23q+.061=0$ he obtains $q=-.0054+r$, and by the same process, $r=-.00004853$. Finally, $y=2+.1-.0054-.00004853=2.09455147$.

Newton seems quite aware that his method of approximation may fail. If there is doubt, he says, whether $p=.1$ is sufficiently close to the truth, find p from $6p^2+10p-1=0$, but he does not inquire whether even this latter method will always answer. He gives an explanation of the solution of $y^3-2y-5=0$, but does not develop his method further. No other examples are given.

Wallis does not praise Newton's method over the older; he merely states that it "is very different from that of Vieta, Oughtred, and Harriot, which is commonly received." The words "very different" are in marked contrast to the statements of the modern historians, H. Hankel* and M. Cantor,† who make Vieta's method of approximation appear as almost identical with the procedure given by Newton. The two are not the same. The essential difference lies in the divisors used in finding the successive approximations. If r is the approximation already reached, then Newton uses a divisor which in our modern notation takes the form $f'(r)$; Vieta's divisor may be expressed in the form $|f(r+s_1)-f(r)|-s_1{}^n$, where $f(x)$ is the left side of the equation $f(x)=k$, n the degree of the equation; and s_1 is

* H. Hankel, *Geschichte der Mathematik*, Leipzig, 1874, pp. 369, 370.

† M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Bd. II, Leipzig, 1900, pp. 640, 641.

a unit of the denomination of the digit next to be found. Thus, in Vieta's example, $x^5-5x^3+500x=7905504$, if $r=20$, Vieta's divisor is 878295, while Newton's divisor is 794500. The inaccuracy of Hankel and Cantor consists in attributing to Vieta the same divisor as that used by Newton. This error probably arose from the fact that in some quadratic equations given by Vieta the two divisors happen to be the same in value. The advantage of the Newtonian divisor over that of Vieta lies in the smaller amount of computation usually called for.

The study of Newton's solution of $y^3-2y-5=0$ reveals the fact that his method of approximation is not the same as what is called "Newton's method" in modern text-books. If r is the initial approximation to a root of the equation $f(x)=0$, then $r_1=r-\frac{f(r)}{f'(r)}$ and $r_2=r_1-\frac{f(r_1)}{f'(r_1)}$, are designated in modern manuals as approximation by "Newton's method." But from the standpoint of the practical computor, this procedure is quite different from that of Newton. Newton derives, as we have seen, each successive step p, q, r of approach to the root, from a *new* equation; while in the modern process just described each approach is made by substitution in the *original* equation.

This modification of Newton's process was first effected by Joseph Raphson (1648(?)—1715(?)), a Fellow of the Royal Society, who published in 1690 in London, a booklet, bearing the title, *Analysis aequationum universalis.* A second edition, with an appendix, appeared in 1697.*

DeMorgan remarks that Raphson does not mention Newton; Raphson evidently considered the difference sufficient for his method to be classed independently. He does not mention Vieta. In our opinion, Raphson's paper is of much greater historical interest than is ordinarily attached to it. Its importance lies in the modification which it makes of Newton's method. Lagrange recognized the modified process as "plus simple que celle de Newton;" and it has now supplanted Newton's method.

In view of the facts it is doubtful whether the method of approximation described by Raphson should be named after Newton alone. In the first place, the processes used by Newton, though not identical to that of Vieta, resembles it. Newton merely simplified the divisor used. Hence the honor of invention falls largely on Vieta. In the second place, Newton did not develop his method further than simply to solve the cubic $y^3-2y-5=0$. That Raphson worked independently of Newton we doubt. But Raphson's version of the process has been accepted as an improvement. It would seem, therefore, that the "Newton-Raphson method" would be a designation more nearly representing the facts of history than is "Newton's method."

We proceed to the description of some details in Raphson's publica-

* A copy of the edition of 1697 is in the New York Public Library. As this copy is in the reference department, I was not able to borrow it through a library loan. My knowledge of Raphson's book is derived mainly from the extracts given in the Latin edition of Wallis' *Algebra*, London, 1693, and the extracts given in an appendix, by Maseres, to W. Frend's *Principles of Algebra*, London, 1796, pp. 467, 468.

tion. We have pointed out that the form now so familiar, $\frac{f(r)}{f'(r)}$, was not used by Newton, but was used by Raphson. To be sure, Raphson does not use this notation; he writes $f(r)$ and $f'(r)$ out in full as polynomials. Nor does Raphson find the first derivative by the rule of the calculus; his operations are purely algebraical. He gives "canons" or "theorems" for the forms of the fractions, in which he writes down the polynomials $f(r)$ and $f'(r)$ for each equation up to that of the tenth degree inclusive. From the standpoint of algebraic notation it may be of interest to quote his expressions for one of these, say the quintic. Taking 1, b, c, d, ... as the coefficients of the given equation and g as the value of the approximation already reached, he writes for the quintic the following "canon:"

"*Pro potestate Quinta*

$$\left.\begin{array}{rr} g\,g\,g\,g\,g & 5\,g\,g\,g\,g \\ b\,g\,g\,g\,g & 4\,b\,g\,g\,g \\ c\,g\,g\,g & 3\,c\,g\,g \\ d\,g\,g & 2\,d\,g \\ f\,g & f \end{array}\right\} x''$$

The terms in the left hand column represent $f(g)$, those in the right hand column $f'(g)x$. With admirable consistency, Raphson gives the "canon," "pro potestate decima," by writing out each of the ten factors in g^{10}.

Newton and Raphson explained their methods only in connection with rational integral algebraic equations. The extension of the Newton-Raphson method to irrational and transcendental equations appears to have been made for the first time by Thomas Simpson, in his *Essays...on Mathematics*, London, 1740, p. 81. He does not mention Newton and Raphson, and calls his procedure a "new method."

Nearly all eighteenth century writers and most of the early writers of the nineteenth century carefully discriminate between the method of Newton and that of Raphson. Then appear writers like Euler, Laplace, Lacroix and Legendre, who explain the Newton-Raphson process, but use no names. Finally, in a publication of Budan in 1807,* in those of Fourier of 1818 and 1831,† in that of Dandelin in 1826,‡ the Newton-Raphson method is attributed to Newton. The immense popularity of Fourier's writings led to the universal adoption of the misnomer "Newton's method" for the Newton-Raphson process.

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* *Nouvelle méthode pour la résolution d. équat. num.*, Paris, 1807, p. 78.

† *Bulletin d. sciences par la société philomatique de Paris*, année 1818, p. 61; Fourier, *Analyse des équations déterminées*, 1831, pp. 169, 173, 177, etc.

‡ *N. mémoires d. l' académie r. d. sciences et belles-lettres de Bruxelles*, T. 3, 1826, pp. 28, 29.

ON CERTAIN SPACE GENERALIZATIONS.

By REV. ALAN SPENCER HAWKESWORTH, University of Pittsburgh.

By its essential concept, a rectangular figure of the nth dimension must be ultimately bounded by $2n$ figures of the next lower, $(n-1)$st dimension; must have 2^n vertices; and from every such vertex n mutually perpendicular lines must spring. On the other hand, each such line has two bounding points, and hence the figure must have $\frac{1}{2}n$ times as many edges as points, or a total of $n(2^{n-1})$.

In like manner, since from every edge starts a square of four edges in two dimensional space, from every edge two squares in three dimensions, three in four dimensions, and $n-1$ squares in n dimensions; it follows that the number of said squares in our n dimensional rectangle must be

$$\frac{(n)\ (n-1)}{2!}2^{n-2}.$$

By similar reasoning, the number of bounding cubes must be

$$\frac{(n)\,(n-1)\,(n-2)}{3!}2^{n-3}.$$

Hence, to sum up, the a-dimensional boundaries of any rectangular figure in n dimensions, must be $\frac{n!}{(n-a)!(a)!}\cdot 2^{n-a}$.

Similar reasoning shows that figures in n dimensions, analogous to the equilateral triangle, and the tetrahedron, will be bounded by $n+1$ figures of the next lower $(n-1)$st dimension; and have $n+1$ vertices, $\frac{(n+1)\,(n)}{2!}$ edges, $\frac{(n+1)\,(n)\,(n-1)}{3!}$ triangular faces, $\frac{(n+1)\,(n)\,(n-1)\,(n-2)}{4!}$ bounding tetrahedrons; and in brief, $\frac{(n+1)!}{(n-a)!\ (a+1)!}$ boundaries of the ath dimension.

From which it follows that the numerical values of the boundaries must rise and fall symmetrically to a medial value; the number $n+1$ of the vertices equalling that of the last $n-1$ boundaries, as stated; the edges and the $n-2$ boundaries each being $\frac{(n+1)\ (n)}{2!}$; the planes and the $n-3$ boundaries each being $\frac{(n+1)\,(n)\,(n-1)}{3!}$, and so on.

Note that the formulae hold true, even when the apparently absurd

question is asked concerning the number of higher dimensional boundaries in a lower dimensional figure, that is, when a is taken greater than n. For the negative and fractional, or reciprocal, result, which we then obtain, as for example, that there are $-\frac{1}{6}$ cubes in a square, means simply that, reversely and reciprocally, there are $+6$ squares in every cube.

Lastly, in extension of Euler's theorem that in all three-dimensional rectilinear solids the sum of their vertices and surfaces is equal to their edges plus 2, we can deduce this general formula, namely: In every even $(2n)$-dimensional rectilinear figure the algebraic sum of its boundaries, taken in their sequence alternately plus and minus, is always zero, while the similar summation of those of every odd $(2n+1)$-dimensional figure is always $+2$.

The proof of this is as follows: First, for rectangular and tetrahedroidal figures.

Any rectangular figure of the nth dimension can be generated by the rectangular movement in n dimensional space of the corresponding figure in the next lower $(n-1)$st dimension. Through this movement the vertices, edges, squares, and cubes, etc., of the generating figure must be given in duplicate; while each vertex must trace an extra edge, each edge a square, each square a cube, and so on. So that letting v, e, p, s, etc., stand for the vertices, edges, planes, surfaces, etc., of our generating $(n-1)$st dimensional figure; and v_1, e_1, p_1, s_1, etc., for those of the generated n dimensional figure, we have always $v_1=2v$; $e_1=2e+v$; $p_1=2p+e$; $s_1=2s+p$; etc.

Now let us represent by l the number and character of the ultimate $(n-2)$ boundaries of our $(n-1)$ figure, which figure we will also call an m. When the last two boundaries of our generated (n) figure must plainly be

$$l_1=(2l+\ldots) \text{ and } m=(2m+l).$$

The algebraic sum of the boundaries of our m figure of $(n-1)$ dimensions, with alternate terms taken $+$ and $-$, will be $+v-e+p-s+\ldots\ \ldots\mp l$, where the sign of the last term l will be $-$ or $+$ according as m is an even or an odd dimensional figure.

Similarly then the sum of the boundaries, taken alternately $+$ and $-$, of our generated n figure will be $+v_1-e_1+p_1-s_1+\ldots\mp l_1\pm m_1=+2v-(2e+v)+(2p+e)-(2s+p)+\ldots\mp(2l+\ldots)\pm(2m+l)=+v-e+p-s+\ldots\mp l\pm 2m$, the sign of the last term $2m$ being again $+$, when the n figure is an odd dimension, but $-$, when it is of an even dimension.

Therefore in any two successive rectangular figures of the $(n-1)$st and nth dimension respectively, the algebraic sum, as above, of the successive boundaries of n, taken alternately $+$ and $-$, will be always greater than those of $(n-1)$, by $+2$, if $(n-1)$ be even and n odd. But if conversely, $(n-1)$ is odd and n even, then the said boundaries of n are always less by -2 than those of $(n-1)$.

Beginning now with the even dimensional square, $+v-e$ is zero, and

in the odd dimensional cube $+v-e+p$ is $+2$; which we know to be correct. Hence the hypothetical fourth, sixth, eighth, etc., dimensional rectangles must always have this summation of their boundaries equal to zero, while in the case of the fifth, seventh, ninth, etc., dimensional rectangles this sum is $+2$.

Taking up now the higher dimensional figures, analogous to the equilateral triangle and the regular tetrahedron, again let $+v-e+p-s+\ldots\mp l$ be the summation of the vertices, edges, planes, solids, etc., up to the limiting l boundary of such an $(n-1)$st dimensional figure, which we will, as before, call m, the sign of the final boundary being once more $-l$, when m is an even dimensional figure, but $+l$ when it is odd; while $+v_1-e_1+p_1-s_1+\ldots\mp l_1\pm m_1$ will be, as before, the summation of the boundaries of the next higher n figure, the signs being $-l_1+m_1$ when n is of an odd dimension, but $+l_1-m_1$ when n is even.

Then the law is $v_1=v+1$; $e_1=e+v$; $p_1=p+e$; $s_1=s+p$; $\ldots m_1=m+l$. Hence $+v_1-e_1+p_1-s_1+\ldots\mp l_1\pm m_1=+(v+1)-(e+v)+(p+e)-(s+p)+\ldots\mp(l+\ldots)\pm(m+l)=+1\pm m=+2$ or zero.

So that here also the summation of the boundaries, taken alternately $+$ and $-$, is always equal to zero, when n is an even and $(n-1)$ an odd dimensional figure, but is equal to $+2$ when n is odd and $(n-1)$ even.

This then fully proves the theorem for at least rectangular and tetrahedroidal figures of the nth dimension. And that it is equally valid for any rectilinear figure whatsoever in n dimensions can also be shown.

For alike in cuboid, tetrahedroid, and the analogue of the tetrakaidecagon in the fourth dimension, from any corner must stretch four edges, six planes and four solids. So that, truncating a corner in any one of the figures, we will add four new vertices and lose an old one; or will add $+3$ vertices in all. We will create six new edges and three new planes but no new solids. hence our new figure adds $+3v-6e+3p=0$ and the previous summation of the untruncated figure is not affected,

But a still more general proof, valid in all dimensions is the following: By the dimensional concept, from every one-dimensional line must stretch two plane faces and one solid and from each plane face a solid, in all three dimensional figures. In those of the fourth dimension, every edge must touch three plane faces and three solids. While every face must bound and be common to two of the bounding solids. And thus in any n-dimensional figure each line must touch $(n-1)$ plane faces. Each face must touch $(n-2)$ solids and every solid must have $(n-3)$ fourth dimensional boundaries and so on, until at last each $(n-2)$ boundary must touch $(n-[n-2])=+2$ of the $(n-1)$ boundaries of our n-dimensional figure.

Therefore taking, say, any fourth-dimensional figure, let us build it up step by step from its component l solids. And let us represent the vertices, edges and planes of our first solid by $+v_1-e_1+p_1$; those of the second by $+v_2-e_2+p_2$, and so on.

Take the first solid $+v_1-e_1+p_1=+2$, and join it to the second solid, so that an identical face in each coincides. Now were we dealing with three-dimensional space, we should evidently have lost two plane faces, each having $v=e$; the said faces passing into the interior of our new composite solid figure, which solid figure has yet $+v-e+p=+2$.

But we are concerned with four-dimensional space, wherein all the planes, etc., of our first solid remain unaltered, and where the second and added solid loses but *one* face, with its equal vertices and edges, by coalescing with the similar face in the first solid. Hence now $v_2-e_2+p_2=+3$.

Similarly, adding the third bounding solid, in place of four planes disappearing, as they would in three-dimensional space, we lose by coalescing but two, and thus $+v_3-e_3+p_3=+4$.

And so on, until we come to the last solid but one, $(l-1)$. When as before, $+v_{l-1}-e_{l-1}+p_{l-1}=+l$.

But upon adding the last l bounding solid, no new boundaries appear. All of its vertices, edges, and planes coalescing with those already existing in our built up figure, so that $+v_l-e_l+p_l=+l$=the solid boundaries, by definition. And hence $+v-e+p-s=0$, for any and all fourth-dimensional figures.

And the same would be true, were a fifth-dimensional figure given, with m bounding fourth-dimensional figures. For, taking the first of the said m figures, we have $+v_1-e_1+p_1-s_1=0$. Adding the second, and thereby losing but one solid, in place of two, we have $+v_2-e_2+p_2-s_2=-1$. The third figure, in like manner, gives us $+v_3-e_3+p_3-s_3=-2$. Until finally, $+v_{m-1}-e_{m-1}+p_{m-1}-s_{m-1}=-(m-2)=+2-m=+v_m-e_m+p_m-s_m$. Or $+v-e+p-s+m=+2$.

And quite similarly, for any dimension. The odd dimensions always adding $+2$ to the zero summation of the previous even figure, and conversely, the even always taking -2 from the $+2$ summation of the previous odd, as we similarly proved for rectangular and tetrahedroidal figures.

Lastly, were we to count the n-dimensional figure itself as its own boundary, and thus add -1 or $+1$ to our previous summation, according as n is odd or even, then we obtain the still more simple rule that any and all such summations always give $+1$, whatever may be the rectilinear figure.

In this case the binomial development of $(2-1)^n$ will give us both the sequence and the summation of the boundaries of a rectangular n-dimensional figure; while, in a similar way, the symmetrical sequence and summation of those of an n-dimensional tetrahedroidal figure can be represented by $-(1-1)^{n+1}$, omitting the first term of the binomial development.

NOTE ON CAUCHY'S INTEGRAL TEST.

By M. B. PORTER, University of Texas.

It does not seem to have been generally noticed that series which consist of a monotonic decreasing sequence of terms of the same sign lend themselves to calculation as readily as those whose signs alternate provided the integral $\int_n^\infty u_n dn$ is sufficiently simple. We have at once, if the terms are all positive,

$$\int_n^\infty u_n dn \leqq u_n + u_{n+1} + u_{n+2} + \ldots \text{etc.} \leqq u_n + \int_n^\infty u_n dn.$$

Thus if to the sum of the first $n-1$ terms we add $\frac{u_n}{2} + \int_n^\infty u_n dn$, our approximation will not differ from the sum of the series by more than $u_n/2$, that is, by less than half of the first term we omit.

Thus to calculate the series $\sum_1^\infty n^{-2}$ the correction after $n-1$ terms is $\frac{1}{2n^2} + \frac{1}{n}$, and the calculated value is in error not more than $\frac{1}{2n^2}$. This fact can be used to calculate with a minimum of labor such slowly convergent series as those represented by the logarithmic scales of DeMorgan.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

346. Proposed by E. B. ESCOTT, Professor of Mathematics, University of Michigan.

Solve completely by quadratics:

$$\frac{2x}{1-x^2} = y; \quad \frac{2y}{1-y^2} = z; \quad \frac{2z}{1-z^2} = w; \quad \frac{2w}{1-w^2} = x.$$

Solution by the PROPOSER.

Let $x = \tan\phi$, then $y = \tan 2\phi$, $z = \tan 4\phi$, $w = \tan 8\phi$, $x = \tan 16\phi = \tan\phi$, $\tan 16\phi - \tan\phi = 0$, or $\frac{\sin 15\phi}{\cos 16\phi . \cos\phi} = 0$.

If $\cos\phi=\infty$, $\tan\phi=\pm i$.

If $\sin 15\phi=0$, $\phi=\frac{n\pi}{15}$ $(n=0, 1, 2, \ldots, 14)$.

$$\therefore x=\pm i,\ 0,\ \tan\frac{\pi}{15},\ \tan\frac{2\pi}{15},\ \ldots,\ \tan\frac{14\pi}{15},$$

$$=\pm i,\ 0,\ \pm\frac{\sqrt{[10+2\sqrt{(5)}]}-\sqrt{(15)}+\sqrt{(3)}}{\sqrt{[30+6\sqrt{(5)}]}+\sqrt{(5)}-1},\ \pm\frac{\sqrt{(15)}+\sqrt{(3)}-\sqrt{[10-2\sqrt{(5)}]}}{\sqrt{[30-6\sqrt{(5)}]}+\sqrt{(5)}+1},$$

$$\pm\frac{\sqrt{[10-2\sqrt{(5)}]}}{\sqrt{(5)}+1},\ \pm\frac{\sqrt{[10+2\sqrt{(5)}]}+\sqrt{(15)}-\sqrt{(3)}}{\sqrt{[30+6\sqrt{(5)}]}-\sqrt{(5)}+1},\ \pm\sqrt{(3)},$$

$$\pm\frac{\sqrt{[10+2\sqrt{(5)}]}}{\sqrt{(5)}-1},\ \pm\frac{\sqrt{(15)}+\sqrt{(3)}+\sqrt{[10-2\sqrt{(5)}]}}{\sqrt{[30-6\sqrt{(5)}]}-\sqrt{(5)}-1}.$$

(These values are taken from Hobson's *Plane Trigonometry.*)

Professor J. Scheffer gave a partial solution, following the method in the above solution. Mr. Spuner sent in a similar solution, with results given in decimals instead of radicals.

347. Proposed by GUSTAVE JACOBSON, A. M., Public Accountant, Chicago, Ill.

A corporation needing some additional capital for a short term of years, issues \$300,000 of debenture bonds carrying 6% interest, and payable 1/5 each year for 5 years. Coupons are attached to the bonds maturing every six months; the bonds are sold at 90 flat. What average rate of interest does the company pay for the money, including interest on interest?

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let $2x$=the rate of interest paid per annum, payable semi-annually; then x=the rate of semi-annual interest, compound. Only \$270,000 was used by the corporation, which was repaid, interest in 10 payments and principal in 5 payments; and they alternate, the first each six months and the second at the end of each year as follows: (1), \$9,000; (2), \$69,000, (3), \$7,200; (4), \$67,200; (5), \$5,400; (6), \$65,400; (7), \$3,600; (8), \$63,600; (9), \$1,800, and (10), \$61,800.

The problem may now be treated, algebraically, as a question in partial payments under the U. S. Court Rule; and we have, after taking out the factor 600, arranging, dropping the dollar sign as follows:

$$450(1+x)^{10}-[15(1+x)^9+115(1+x)^8+12(1+x)^7+112(1+x)^6$$
$$+9(1+x)^5+109(1+x)^4+6(1+x)^3+106(1+x)^2+3(1+x)]=103.$$

Observe that as arranged the first member has 2 terms, and their difference

=103, As the debenture was discounted, the rate realized on the investment must be greater than 3% as a semi-annual rate. We try 5% and find it too small; we then try 5.0275%, with $(1+x)=1.050275$, and find the first member=102.9337, which is too small; we then try $(1+x)=1.050285$ and obtain the value 102.9531, still too small. By double position we have a corrected value and $(1+x)=1.050309$ and obtain 103.0304, which is now too large. By double positon we have $(1+x)=1.050291$, and obtain 102.9937, again too small. By double position again we have $(1+x)=1.0502941$ and obtain 103.0033, now too large. We now again use double position and obtain, $(1+x)=1.05029269$, this will give a value which will differ by less than 1% of a mill. We now have for a good approximation a semi-annual rate, 5.029269%; or the rate per annum payable semi-annually, 10.058538%.

Also solved by S. A. Corey who got for a result 10.28 per cent.; A. H. Holmes, whose result was 10.07 per cent., and the proposer, whose result was 10.0532 per cent. These contributors used practically the same method of solution. Mr. Spuner also sent in an excellent solution of 345, which reached us too late for credit in the January number.

CALCULUS.

298. Proposed by C. N. SCHMALL, New York City.

Prove, by calculus, that if two regular polygons have equal perimeters, that which has the greater number of sides has the greater area.

Solution by E. L. SHERWOOD, Shady Side Academy, Pittsburg, Penn., and the PROPOSER.

In any regular polygon, let $2p$=the perimeter, n=number of sides.

$\therefore \frac{2p}{n}$=each side, and $\frac{p}{n\tan(\pi/n)}$=the apothem.

$$\frac{p^2}{n\tan(\pi/n)}=\frac{p^2}{n}\cot\frac{\pi}{n}=\text{area}\ldots(1).$$

Put $u=n\tan\frac{\pi}{n}\ldots(2)$; hence the area will be a maximum when u is a minimum.

From (2) we have

$$\frac{du}{dn}=\tan\frac{\pi}{n}-\frac{\pi}{n}\sec^2\frac{\pi}{n}=\tan\phi-\phi\sec^2\phi\ (\text{where}\ \frac{\pi}{n}=\phi)=\sin\phi\ \sec\phi-\phi\sec^2\phi$$

$$=\tfrac{1}{2}(2\sin\phi\ \sec\phi-2\phi\sec^2\phi)=\tfrac{1}{2}\sec^2\phi\left(\frac{2\sin\phi}{\sec\phi}-2\phi\right)=\tfrac{1}{2}\sec^2\phi(\sin2\phi-2\phi),$$

$$=\tfrac{1}{2}\sec^2\frac{\pi}{n}\left(\sin\frac{2\pi}{n}-\frac{2\pi}{n}\right).$$

This expression is negative for $n>2$ and therefore decreases as n increases. Hence the area increases as n increases. Q. E. D.

NOTE. Equating the derivative to zero, we have

$$\tan\frac{2\pi}{n}-\frac{2\pi}{n}=0.$$

This equation is true only for very small values of the angle. In fact, it approximates to exactitude as the angle $2\pi/n$ approaches zero. Hence, the area of the polygon opproaches its maximum value (as a limit) as n approaches ∞; it reaches its maximum when $n=\infty$. In that case the polygon becomes a circle, which is, therefore, the greatest of all isoperimetric plane figures. In a similar manner, it may be shown that of all plane figures of given area the circle has the least periphery.

Also solved by J. E. Sanders, V. M. Spunar, J. Scheffer, and S. G. Barton.

299. Proposed by JOSEPH V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

A cow is pasturing outside a circular field containing 10 acres. What length of rope will allow her to graze over exactly two acres?

Solution by J. SCHEFFER, A. M., Hagerstown, Md.; J. E. SANDERS, Columbus, Ohio, and S. G. BARTON, Ph. D., Clarkson School of Technology.

It is assumed that the cow is tethered at a point in the circumference of the field.

It is evident that the area of the pasture consists of a semi-circle, of radius equal to the length of the rope and two portions contained between the circular boundary of the field and the involute of the circle.

Let a be the radius of the field, l the length of the rope. The intrinsic equation of the involute of a circle is

$$s=a\frac{\psi^2}{2}.$$

Considering the element of area contained between this involute and its evolute, that is, the circle, and two radii of curvature, we have

$$dA=\rho\frac{ds}{2},\quad \rho=\frac{ds}{d\psi}=a\psi;\quad \therefore s=\frac{\rho^2}{2a},\quad ds=\frac{\rho}{a}d\rho.\quad \therefore dA=\frac{\rho^2}{2a}d\rho.$$

The limits of ρ for area involved are evidently 0 and l.

$$\therefore A=\int_0^l \frac{\rho^2}{2a}d\rho=\frac{l^3}{6a}.\quad \text{Hence area}=2\text{ acres}=\frac{\pi l^2}{2}+\frac{l^3}{3a}.$$

Using data of problem, $a=372.4$ feet. $.0009l^3+1.5708l^2=87120.$ $l=221.9$ feet.

Solved in a slightly different way by V. M. Spunar.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

175. Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Nebraska.

Show (a) that $[(2n)!]/[(n+1)!n!]$ is an integer, and (b) that $[(2a)!(2b)!]/[a!b!(a+b)!]$ is an integer.

Solution by G. E. WAHLIN, Ph. D., University of Illinois.

(a) To show that $\frac{(2n)!}{(n+1)!(n!)^2}$ is an integer.

$$\frac{(2n)!}{(n+1)\ (n!)^2}=\frac{(2n)!}{(n+1)!\,n!}=\frac{2n(2n-1)\ (2n-2)\ldots(n+1)}{(n+1)!}.$$

$I=\frac{2n\,(2n-1)\ldots(n+1)}{n!}$ is an integer, being the quotient of n successive integers by $n!$.

For the same reason, $I'=\frac{(2n+1)\,(2n)\,(2n-1)\ldots(n+1)}{(n+1)!}$ is an integer.

But $I'=\frac{2n+1}{n+1}I$; and since $2(n+1)-(2n+1)=1$,

the numerator and denominator are relatively prime in $\frac{2n+1}{n+1}$, and therefore since $I.\frac{2n+1}{n+1}$ is an integer, I must be divisible by $n+1$, or $\frac{I}{n+1}$ is an integer.

But $\frac{I}{n+1}=\frac{2n\,(2n-1)\ldots(n+1)}{(n+1)!}=\frac{2n!}{(n+1)\ (n!)^2}.$

(b) To show that $\frac{(2a)!(2b)!}{a!b!\ (a+b)!}$ is an integer.

It is evident that the given expression is equal to

$$\frac{(a+1)\,(a+2)\ldots.2a\,(b+1)\,(b+2)\ldots.2b}{(a+b)!}\ldots(1),$$

and it is therefore sufficient to show that this is an integer.

Let p be any prime less than or equal to $a+b$, and n any integer such than $p^n \leq a+b$. Moreover, let $a=kp^n+c$ and $b=k'p^n+c'$, where c and c' are both less than p^n.

We shall determine the number of multiples of p^n in the sequence

$$a+1,\ a+2,\ \ldots,\ 2a \ldots (2).$$

Consider first the case when $c=p^n-1$. Then $a+1=(k+1)p^n$ and $2a=2kp^n+2p^n-2=(2k+1)p^n+p^n-2$, and hence $(k+1)p^n$ is, in this case, the first multiple of p^n in the sequence, and $(2k+1)p^n$ is the last. If we let v_n be the number of such multiples in the sequence, since they form an arithmetic progression, we have

$$(2k+1)p^n=(k+1)p^n+(v_n-1)p^n.$$

Solving for v_n, we get $v_n=k+1$.

If $\frac{p^n-1}{2}<c<p^n-1$, then $a+1=kp^n+c+1$, and $2a=2kp^n+2c=(2k+1)p^n+d$, $d<p^n$. Therefore, in this case again, the first and last multiples of p^n in (2) are, respectively, $(k+1)p^n$ and $(2k+1)p^n$. As before, therefore, $v_n=k+1$.

If $c \leq \frac{p^n-1}{2}$, then $a+1=kp^n+c+1$, and $2a=2kp^n+2c$, $2c<p^n$, and in this case the first and last multiples of p^n in the sequence are, respectively, $(k+1)p^n$ and $2kp^n$. As above, we then find that $v_n=k$. We have, therefore, that

$$v_n=k+1 \text{ when } \frac{p^n-1}{2}<c \leq p^n-1; \quad v_n=k \text{ when } c<\frac{p^n-1}{2}.$$

In the same way, letting v_n' be the number of multiples of p^n in the sequence, $(b+1)$, $(b+2)$, ..., $2b$, we find that

$$v_n'=k'+1 \text{ when } \frac{p^n-1}{2}<c' \leq p^n-1; \quad v_n'=k' \text{ when } c' \leq \frac{p^n-1}{2}.$$

If we now let m be the largest integral exponent for which $p^m \leq a+b$, since all multiples of p^n are also multiples of the lower powers of p, it is not difficult to see that $v_1+v_2+\ldots+v_m$ is the power to which p enters as a factor in the product $(a+1)(a+2)$, ..., $2a$, and $v'_1+v'_2+\ldots+v'_m$ is the expo-

nent of p in the product $(b+1)(b+2), \ldots, 2b$. Hence the power of p in the numerator of (1) is

$$s=v_1+v_2+\ldots+v_m+v_1'+v_2'+\ldots v_m'.$$

We shall next turn our attention to the sequence 1, 2, 3, ..., $a+b$. In this the first multiple of p^n is p^n and as $a+b=(k+k')p^n+c+c'$ the last is $(k+k'+1)p^n$ or $(k+k')p^n$, according as $c+c'$ is greater than or less than p^n, and if we let v_n'' be the number of such multiples in this sequence we have

$$v_n''=k+k'+1 \text{ when } c+c'>p^n; \quad v_n''=k+k' \text{ when } c+c'<p^n.$$

Hence, as before, the exponent of p in the product $(a+b)!$ is

$$t=v_1''+v_2''+\ldots+v_m''.$$

But for $c+c'$ to be greater than p^n, at least one of the quantities c and c' must be greater than $\frac{p^n-1}{2}$, and it is therefore easily seen that $v_n'' \leq v_n+v_n'$ and hence $t \leq s$.

From the condition imposed on p, this may be any prime factor of $(a+b)!$, and hence the numerator of (1) contains this prime with an exponent not less than its exponent in the denominator, and (1) is therefore an integer.

A proof of the above theorem is also found in *Die Elemente der Zahlen Theorie*, p. 37, by Bachmann.

178. **Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.**

Find a formula which gives all the integral solutions prime to 5 of the congruence $y^2+z^2 \equiv 0 \pmod{5^4}$.

Solution by the PROPOSER.

Since $y^2 \equiv 4z^2 \pmod 5$, $y \equiv \pm 2z \pmod 5$. Hence $y=5r \pm 2z$. Then

$$y^2+z^2=5(5r^2 \pm 4rz+z^2) \equiv 0 \pmod{5^4},$$
$$5r^2 \pm 4rz+z^2 \equiv 0 \pmod{5^3}.$$

Since z is prime to 5, the latter gives $\pm 4r+z \equiv 0 \pmod 5$, whence $r \equiv \pm z \pmod 5$. Thus $r=5s \pm z$. Substituting in $5r^2, \ldots, \equiv 0$, we get $10z^2 \pm 70sz$

$\equiv 0 \pmod{5^3}$. Hence $z \pm 7s \equiv 0 \pmod{5^2}$, so that $z = 25t \mp 7s$. Now $y = 5r \pm 2z = 25s \pm 7z = \pm 175t - 24s$. The only solutions prime to 5 are thus

$$y = \pm 175t - 24s, \quad z = 25t \mp 7s.$$

Conversely, these values satisfy the given congruence.

Problems and solutions for this department should be sent to Dr. Wahlin, Urbana, Ill.

PROBLEMS FOR SOLUTION.

ALGEBRA.

351. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve, $y^2 + yz + z^2 = a^2 \ldots (1)$.
$z^2 + zx + x^2 = b^2 \ldots (2)$.
$x^2 + xy + y^2 = c^2 \ldots (3)$.

352. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve the equations, $x^3 = -8y + 24 \ldots (1)$.
$y^3 = -8x + 24 \ldots (2)$.

GEOMETRY.

381. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the number of diagonals of a *complete* polygon of n sides.

382. Proposed by PROF. R. C. ARCHIBALD, Brown Uniuersity, Providence, R. I.

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length and directed to the opposite corner. "Euclidean constructions" are particularly desired.

383. Proposed by S. A. COREY, Hitsman, Iowa.

Let $ABCDE$ be a pentagon, plane, or gauche, with sides AB, BC, CD and DE, of lengths, w, x, y, and z, respectively. Construct four other pentagons, $AB_1C_1D_1E_1$, $AB_{11}C_{11}D_{11}E_{11}$, $AB_{111}C_{111}D_{111}E_{111}$, and $AB_{iv}C_{iv}D_{iv}E_{iv}$, having a common vertex at A, and with four consecutive sides in each parallel to the corresponding consecutive sides, AB, BC, CD, and DE, in $ABCDE$. Further, let the lengths of the sides AB_1, B_1C_1, C_1D_1, D_1E_1, AB_{11}, $B_{11}C_{11}$, $C_{11}D_{11}$, $D_{11}E_{11}$, AB_{111}, $B_{111}C_{111}$, $C_{111}D_{111}$, $D_{111}E_{111}$, AB_{iv}, $B_{iv}C_{iv}$, $C_{iv}D_{iv}$, $D_{iv}E_{iv}$, be $-wW$, xX, yY, zZ, wX, xW, yZ, $-zY$, wY, yW, zX, $-xZ$, wZ, zW, xY, and $-yX$, respectively; the minus sign indicating the reversal of direction of the corresponding side. Prove that $(W^2 + X^2 + Y^2 + Z^2)(w^2 + x^2 + y^2 + z^2) = E_1A_1^2 + E_{11}A_{11}^2 + E_{111}^2A_{111}^2 + E_{iv}A_{iv}^2$.

CALCULUS.

306. Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.

Express in elliptic integrals: $A_\theta = \int_0^\theta \frac{dx}{\sqrt{(1-x^4)}}$; $0<\theta<0$.

307. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology.

The maximum value is not necessarily the greatest value of a function. Show why we take the maximum value as the greatest value in practical problems in maxima and minima. Likewise minimum for least-value.

MECHANICS.

259. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A uniform beam of the weight W, rests on a horizontal plane, and leans against a vertical wall, but so as *not* to lie in a vertical plane. Denoting the pressure upon the horizontal and vertical planes, respectively, by x and y, the coefficients of friction respectively, by μ and μ'; the angle which the perpendiculars from the foot of the beam upon the intersection of both planes makes with the beam by ϕ; the angle which this perpendicular makes with the direction of the friction peg by ξ; and the angle, which the projection of the beam upon the vertical wall makes with the vertical line, by ψ. To prove: $(1+\mu^2)\mu'^2\sin^4\psi-[1+\mu'^2+4\mu^2\mu'^2]\sin^2\psi+4\mu^2\mu'^2=0$; $\tan\xi=\mu'\cos\psi$, $\tan\phi=\mu'\cot\psi$, $x=\frac{\mu\cos\xi}{1+\mu\mu'\sin\psi\cos\xi}W$, $y=\frac{1}{1+\mu\mu'\sin\psi\cos\xi}W$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

181. Proposed by V. M. SPURAR, M. and E. E., Chicago, Ill.

If $2n+1$ is an odd prime p, $(2n)!\equiv(-1)^n 2^{4n}(n!)^2 \pmod{p^2}$.

182. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Find two general solutions in integers of the equation $x^2=616318177y-1$.

148. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find all the multiply perfect numbers of n different prime factors and of multiplicity $n-1$.

153. Proposed by LLOYD HOLSINGER, A. B., 227 Fredonia Avenue, Peoria, Ill.

If we represent by (k, l) the greatest common divisor of k and l, and by $\phi(k)$ the number of integers prime to k and not greater than k, we have

$$\begin{vmatrix} (1,1) & (1,2) & (1,3) & \dots & (1,n) \\ (2,1) & (2,2) & (2,3) & \dots & (2,n) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (n,1) & (n,2) & (n,3) & \dots & (n,n) \end{vmatrix} = \phi(1).\phi(2).\phi(3)\dots\phi(n).$$

157. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for m and n in $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$.

161. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find a solution of $x^4-5x^2+4\equiv 0 \pmod{p.2p+1}$, where both p and $2p+1$ are odd primes.

162. Proposed by L. E. DICKSON, Ph. D., Associate Professor of Mathematics, The University of Chicago.

If p is an odd prime, find the number of incongruent integers x for which x^4+2ex^2+f is a quadratic residue of p.

166. Proposed by H. S. VANDIVER, Bala, Pa.

Eliminate any five of the seven quantities h, j, k, l, m, r, s, from:

$$j+m+r=k+l+s,$$
$$h=3(r+s)-(n+1),$$
$$j+m+2r+s=n,$$
$$hr+j^2+km+ms=jk+mr+s^2+rs,$$
$$ks+jl+rs+r^2=sh+jk+l^2+rk,$$
$$hr+jl+ks+ls+km=jm+mr+2rs+rj.$$

167. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Prove that $\Pi\frac{p+(-1)^{\frac{1}{2}(p+1)}}{p-(-1)^{\frac{1}{2}(p+1)}}=2$, the consecutive values of p being the natural odd primes in order.

NOTES AND NEWS.

At Princeton University Dr. G. D. Birkhoff, assistant professor, and Dr. William Gillespie, preceptor, have been promoted to full professorships in mathematics. S.

Dr. Anna J. Pell has been elected to an instructorship in mathematics at Mount Holyoke College for the year 1911-1912. She is at present conducting courses at Armour Institute, Chicago, during the illness of her husband, Professor Alexander Pell. S.

Professor E. R. Hedrick, of the University of Missouri, has leave of absence for eight months and will spend the time in travel and study abroad. S.

Professor E. B. Van Vleck, who has been on leave of absence from the University of Minnesota since June, 1910, has returned to resume his duties for the second semester. S.

Dr. H. B. Smith has been appointed Instructor of Mathematics at the University of Pennsylvania, for the ensuing term, to fill the vacancy caused by the temporary absence of Professor Evans. C.

The Department of Superintendents of the National Education Association met at Mobile, Alabama, on February 23, 24, 25, 1911. The annual meeting of the Association is to be held in San Francisco early in July. S.

Professor Gilbert A. Bliss, of the University of Chicago, who during the autumn of 1910 was traveling in Japan and other eastern countries, is now in Paris. He will return to the University of Chicago in Aprill, 1911, having completed the journey around the world. S.

Teachers' College, Columbia University, New York City, has recently issued a bibliography of mathematical works suitable for high school and normal school libraries. This is prepared by Professor David Eugene Smith and Professor Clifford Brewster Upton. Teachers who may care for this bibliography may obtain it gratis by writing to the Secretary of Teachers' College, Columbia University. S.

The first of a series of articles, later to be issued in book form, by Dr. George Bruce Halsted, appeared in February's *Open Court* under the title: The Prehuman Contributions to Arithmetic. Dr. Halsted will, in these articles, transmute the dry bones of arithmetic into a fascinating serial. A commendatory review of the first of this series of articles is given in the *Chicago Tribune* of February 9. F.

The Chicago Section of the American Mathematical Society has completed its fourteenth year, having held since its organization in 1897, twenty-eight regular meetings, at which four hundred and eighty-one papers have been presented, or an average of about eighteen papers per meeting. The first conference on the desirability of organizing such a section was called on December 31, 1896, and was signed by twenty-eight members of the Society, and the conference was attended by seventeen members. At this conference fourteen papers were read and at the second conference in April, 1897, ten papers were read. At the latter time the Chicago Section was formally established. S.

We are grieved again to record the death of another of our loyal subscribers and contributors, Joseph Carter Corbin, whose death occurred January 9, 1911, from heart failure. The following brief sketch of his life was furnished by his daughter Louise: Mr. Corbin was born in Chillicothe, Ohio, March 26, 1833. He was the eldest son of William and Susan C. Corbin; graduated from Ohio University in 1853, (A. M. from the same); married September 11, 1866, to Mary J. Ward, at Cincinnati; taught school in Louisville, Kentucky, about twelve years, assisting his brother-in-law, Rev. Henry Adams; later was employed by the Bank of Ohio Valley in Cincinnati; while there he served two years on the Board of Education, and edited *The Citizen* seven years. Moved to Little Rock, Arkansas, in 1870, and edited *The Republican* for General Powell Clayton; State Superintendent of Public Instruction in 1873; Principal of school at Jefferson City, Missouri, 1874-76; organized Branch Normal College at Pine Bluff, Arkansas, 1876, and was President 1876-1902; President Ouachita Baptist College, Camden, Arkansas, 1902 to January, 1904; at the request of the Board of Education of Pine

Bluff, he resigned and became Principal of Merrill School and served until the close of the school year, May, 1910. He was actively engaged in Masonic work and in conducting a private correspondence school at the time of his sudden death. He was the father of five children, two of whom, William H. and Louise M., survive him. In politics a Republican; in religion a Baptist; and a thirty-third degree Mason. He was a subscriber and contributor to the MONTHLY from its beginning.

BOOKS.

Diophantus of Alexandria. A Study in the History of Greek Algebra. By Sir Thomas L. Heath, K. C. B., Sc. D., sometime Fellow of Trinity College, Cambridge, England. Second edition, with a supplement containing an account of Fermat's Theorems and Problems connected with Diophantine Analysis and some Solutions of Diophantine Problems by Euler. Royal 8vo. Cloth, viii+388 pages. Price, 12s. 6d., net. London, England: The Cambridg University Press.

The first edition, the first English translation of Diophantus' *Arithmetica* which appeared in 1885, has long been out of print and thus that interesting and, to many, fascinating treaties had become practically inaccesible to English readers. Inquiries for it at different times suggested to the author the revision of his work done twenty-five years ago and the result is this second edition which conforms to recent discoveries in the history of mathematics. In this new edition, the author has rewritten the whole of the introduction, except the chapters on the editions of Diophantus, his methods of solutions, and the porisms, and other assumptions found in his work; while the chapters on the methods and on the porisms, etc., have been made fuller.

Much attention has been given to Fermat's notes on the various problems, and as much of this interesting material could not be incorporated as foot notes, the author added a supplement in which much attention is given to the theorems of Fermat and in which also is included a number of Euler's remarkable solutions of difficult Diophantine problems.

In its present form, the book will be in great demand by all mathematicians who are interested in this interesting phase of the theory of numbers. F.

College Mathematics Notebook. By Robert E. Moritz, Ph. D., Professor of Mathematics, University of Washington. Price, 80 cents. Ginn & Co.

This note book will be found very convenient and serviceable for the college teacher of mathematics. F.

ERRATA.

On page 241, No. 12, Vol. XVII, December 1910, problem 343 solution, make the following changes:

In line six of the solution for "A on the day x," read "A on the day $x+1$," and in the next line for "B on the day x," read "B on the day $x+1$." On page 243, in line three, for "$(1.05)^1$," read "$(1.05)^0$," and in the next line for "or," read "of," and in the same line for "0.202," read "0.002." In the fourth line from the end of solution for "(0.8583)," read "(0.9583)."

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TESTS OF SYMMETRIC POLYNOMIALS.

By DR. G. A. MILLER, University of Illinois.

In Bôcher's *Introduction to Higher Algebra*, 1907, page 240, the following theorem is stated: *A necessary and sufficient condition for a polynomial to be symmetric is that it be unchanged by every interchange of two variables.* This test becomes quite laborious even when the number of variables is reasonably small; for instance, for six variables there are fifteen distinct pairs of variables so that this theorem requires fifteen trials to prove that a polynomial in six variables is symmetric. In general, the number of trials according to the given theorem is clearly $\frac{n(n-1)}{2}$, and hence this number increases very rapidly with n. In view of this fact, it may be of interest to consider some tests requiring a very much smaller number of trials, especially since the determination of these tests is equivalent to a determination of sets of generating substitutions of the symmetric group of degree n and hence it is a problem which is common to at least two subjects.

If the following well known theorem: *A necessary and sufficient condition for a polynomial to be symmetric is that it be unchanged by every interchange of two variables such that one variable is the same in each of these pairs of variables,* were substituted for the one given above, the number of trials necessary to prove the symmetry of a polynomial of n variables would evidently be reduced to $n-1$, and the individual trials would not involve any more labor than those of the theorem of the first paragraph. Moreover, the proof of the theorem of the present paragraph involves very little more thought than that of the preceeding paragraph. In fact the $\frac{n(n-1)}{2}$ possible transpositions, or interchanges of pairs of variables formed from n variables, can clearly be obtained by transforming the $n-1$, which have a common variable by means of others of this set of $n-1$ transpositions. That every possible transposition on these n variables may be obtained in this manner, results from the following identity: